Differential Geometry IV: General Relativity

14.1 Let (Σ^3, g) be a smooth Riemannian manifold. We will say that (Σ, g) is *asymptotically flat* with n asymptotically flat ends if there exists a compact subset $\mathcal{K} \subset \Sigma$ such that $\Sigma \setminus \mathcal{K}$ has n connected components $\Sigma_1, \dots, \Sigma_n$ and, for each of them, there exists a diffeomorphism $\Phi_i : \Sigma_i \to \mathbb{R}^3 \setminus B_1(0)$ with the following property: In the Cartesian coordinates (x^1, x^2, x^3) associated to this diffeomorphism, the components of the metric g satisfy for any $m \in \mathbb{N}$:

$$\partial^m (g_{ij} - \delta_{ij}) = O(r^{-m-1}) \quad \text{as} \quad r \to +\infty,$$

where $r = \left(\sum_{i=1}^3 (x^i)^2\right)^{\frac{1}{2}}$. For any asymptotically flat end Σ_l, we will define the ADM mass $(M_{ADM})_l$ as the limit (in these coordinates)

$$M_{ADM} = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{S_r} \Big(\sum_{i,j=1}^3 \big(\partial_j g_{ij} - \partial_i g_{jj}\big) N^i \, dA \Big), \tag{1}$$

where S_r is the coordinate sphere of radius r, N is the normal to S_r (with respect to the flat metric) and dA is the volume form on S_r induced by the flat metric.

(a) Show that the value of the ADM mass in each asymptotically flat end is invariant under coordinate transformations of the form $x \to x + c + F(x)$, where $c \in \mathbb{R}^3$ is a constant and $F : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$\partial^m F = O(r^{-m-1}) \quad \text{for all } m \in \mathbb{N}$$

(coordinates in this class are usually called *asymptotically Euclidean*).

(b) Show that the slice $\{t = 0\}$ in the maximally extended Schwarzschild spacetime with mass parameter $M > 0$, equipped with its induced metric, is asymptotically flat with two asymptotically flat ends. Show that the ADM mass of each end is equal to M.

*(c) Let $(\mathbb{R}^3; \bar{g}^{(\epsilon)}, k^{(\epsilon)})$ be a smooth family of initial data sets for the Einstein equations

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\epsilon T_{\mu\nu}$$

with $\epsilon \geqslant 0$, such that $\bar{g}^{(0)}_{ij} = \delta_{ij}$ and $k^{(0)} = 0$ for all $\epsilon \geqslant 0$. Assume that $(\mathbb{R}^3; \bar{g}^{(\epsilon)})$ is asymptotically flat for all $\epsilon \geqslant 0$. Defining $h = \frac{d}{d\epsilon} \bar{g}^{(\epsilon)}\Big|_{\epsilon=0}$ to be the linearization of $\bar{g}^{\epsilon}$ around $\epsilon = 0$, show that

$$\sum_{i,j=1}^3 \Big(-\partial_i^2 h_{jj} + \partial_i \partial_j h_{ij} \Big) = 16\pi T(\hat{n}, \hat{n})$$

(Hint: Compute the linearization of the Hamiltonian constraint equation.) Deduce that, if the energy momentum tensor T satisfies the positive energy condition $T(\hat{n}, \hat{n}) \geqslant 0$, then

$$\frac{d}{d\epsilon} M^{(\epsilon)}_{ADM}\Big|_{\epsilon=0} \geqslant 0.$$

Remark. The above is a special case of the following fundamental result, proven in two different ways by Schoen–Yau (1979) and Witten (1981):

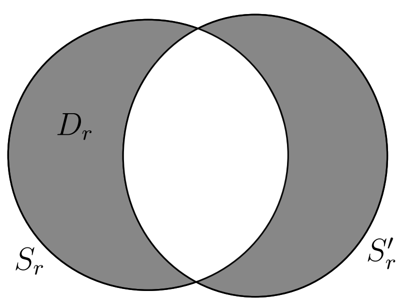
The positive mass theorem: Let $(\Sigma^3, \bar{g}, k)$ be an asymptotically flat initial data set for the Einstein equations for a matter field satisfying the *dominant energy condition* (e.g. vacuum, scalar field, etc). Then the ADM mass of each asymptotically flat end satisfies $M_{ADM} \geqslant 0$, with equality if and only if $(\Sigma, \bar{g}, k)$ is a trivial initial data set, i.e. $\Sigma = \mathbb{R}^3$ and $(\bar{g}, k)$ are the induced metric and second fundamental form of a Cauchy hypersurface of Minkowski spacetime (if $k = 0$, this implies that $\bar{g}$ is the flat Euclidean metric).

Solution.

(a)

Let $\vec{y} \doteq \vec{x} + \vec{c} + F(\vec{x})$ be a coordinate transformation, with F as in the assumption. Note that, when computing the ADM mass M_{ADM} in the two coordinate systems (x^1, x^2, x^3) and (y^1, y^2, y^3) using the formula (1), there are two things that we need to take care of:

1. The expressions for $\partial_i g_{jl}$ are different in the two coordinate systems,
2. The coordinate spheres $S_r = \{(x^1, x^2, x^3) : \sum_{i=1}^3 (x^i)^2 = r^2$ and $S'_r = \{(y^1, y^2, y^3) : \sum_{i=1}^3 (y^i)^2 = r^2$ are possibly different surfaces in $\mathbb{R}^3$ (see also the figure below). We denote by D_r the region between S_r and S'_r.



Let us first see how the coordinate vector fields transform under the coordinate transformation $(x^1, x^2, x^3) \to (y^1, y^2, y^3)$:

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k(\vec{x})}{\partial x^i} \frac{\partial}{\partial y^k} = \left(\delta_i^k + \partial_i F^k\right) \frac{\partial}{\partial y^k} = \frac{\partial}{\partial y^i} + \partial_i F^k \frac{\partial}{\partial y^k}.$$

Note that the coefficient $\partial_i F^k$ above satisfies

$$\partial_{x^i} F^k = O(r^{-2}) \tag{2}$$

(in view of our assumption on F). Then, using the above we can expand and show

$$\partial_j g_{ij} = \partial_{x^j} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \partial_{y^j} g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) + \mathcal{O}(r^{-2}) \qquad \text{as} \quad r \to \infty \tag{3}$$

Note, terms in the expansion above of the form

$$\partial_{y^j}\left(\partial_i F^k\right) \cdot g\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}\right)$$

satisfy $\partial_{y^j}\left(\partial_i F^k\right) = \mathcal{O}(r^{-3})$. To see this, we write

$$\partial_{y^j}\left(\partial_i F^k\right) = \frac{\partial x^m}{\partial y^j} \partial_{x^m} \partial_{x^i} F^k$$

By assumption, we have that $\partial_{x^m}\partial_{x^i} F^k = \mathcal{O}(r^{-3})$ as $r \to \infty$, so we only need to justify that $\frac{\partial x^m}{\partial y^j}$ are bounded functions (i.e. they don't contribute any $r-$growth) as $r \to +\infty$. However, by the inverse function theorem, these functions correspond to the components of the inverse of the Jacobian matrix $\mathbf{J}_y = \frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}$, which we computed before to be

$$\mathbf{J}_y = \mathbf{I} + [dF], \text{ or, in components, } (\mathbf{J}_y)_i^j = \delta_i^j + \partial_{x^i} F^j].$$

Using the asymptotics (2) and the formula $(\mathbf{I} + A)^{-1} = \mathbf{I} - A + O(\|A\|^2)$, we infer the boundedness of $\frac{\partial x^m}{\partial y^j}$ as $r \to \infty$.

Similarly, we have

$$\partial_i g_{jj} = \partial_{x^i} g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^j}\right) = \partial_{y^i} g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j}\right) + \mathcal{O}(r^{-2}) \qquad \text{as} \quad r \to \infty \tag{4}$$

while also, the components of the normal vector to S_r, $N = N^i \frac{\partial}{\partial x^i} = N^i \frac{\partial}{\partial y^i} + h^k \frac{\partial}{\partial y^k}$, with $h^k = \mathcal{O}(r^{-2})$ as $r \to \infty$, for any k.

Let us denote by $f : \Sigma \setminus \mathcal{K} \to \mathbb{R}$ the function which in the (x^1, x^2, x^3) coordinate chart takes the form

$$f(x) = \sum_{i,j=1}^{3} \left(\partial_{x^j}(g(\partial_{x^i}, \partial_{x^j}) - \partial_{x^i}(g(\partial_{x^j}, \partial_{x^j})\frac{x^i}{r}\right)$$

We will similarly denote by $f' : \Sigma \setminus \mathcal{K} \to \mathbb{R}$ the function which in the (y^1, y^2, y^3) coordinate chart takes the form

$$f'(y) = \sum_{i,j=1}^{3} \left(\partial_{y^j}(g(\partial_{y^i}, \partial_{y^j}) - \partial_{y^i}(g(\partial_{y^j}, \partial_{y^j})\frac{y^i}{r}\right),$$

(where $r = \big(\sum_{i=1}^{3}(y^i)^2\big)^{\frac{1}{2}}$). Note that the expression of the ADM mass in the (x^1, x^2, x^3) coordinates takes the form

$$M_{ADM} = \frac{1}{16\pi}\lim_{r\to+\infty}\int_{S_r} f\, dA = \frac{1}{16\pi}\lim_{r\to+\infty}\int_{S_r} f\,\big(1+O(r^{-1})\big)dvol_{\bar{g}}$$

where, in passing to the second equality above, we used the fact that the flat volume form dA and the volume form of the induced metric $dvol_{\bar{g}}$ differ by a factor of the form $1+O(r^{-1})$ (since $g_{ij}-\delta_{ij} = O(r^{-1})$). Similarly, the expression of the ADM mass in the (y^1, y^2, y^3) coordinates gives:

$$M'_{ADM} = \frac{1}{16\pi}\lim_{r\to+\infty}\int_{S'_r} f'\,\big(1+O(r^{-1})\big)dvol_{\bar{g}}.$$

In view of our previous calculations, we have

$$f(x) - f'(x) = O(r^{-3}) \quad \text{at a point } x = (x^1, x^2, x^3) \quad \text{with } r = \big(\sum_{i=1}^{3}(x^i)^2\big)^{\frac{1}{2}}.$$

Therefore, since S_r and S'_r have area $\sim r^2$, we have

$$M_{ADM} - M'_{ADM} = \frac{1}{16\pi}\lim_{r\to+\infty}\left(\int_{S_r} f\,\big(1+O(r^{-1})\big)dvol_{\bar{g}} - \int_{S_r} f'\,\big(1+O(r^{-1})\big)dvol_{\bar{g}}\right) = \frac{1}{16\pi}\lim_{r\to+\infty}\left(\int_{S_r} f\, dvol_{\bar{g}} - \right.$$

We can similarly compute that

$$\|\nabla_x f(x)\| = O(r^{-4}).$$

Noting that, in the (x^1, x^2, x^3) coordinate system, the surface S'_r lies within distance $O(1)$ from S_r (since $S_r = \Big\{(x^1,x^2,x^3) : \sum_{i=1}^{3}(x^i)^2 = r^2\Big\}$ and $S'_r = \Big\{(x^1,x^2,x^3) : \sum_{i=1}^{3}(x^i + c^i + O(r^{-1})^2 = r^2\Big\}$), the domain D_r is contained inside an annulus around S_r of width $O(1)$, and hence has volume $\sim O(r^2)$. Therefore, we can estimate

$$\Big|M_{ADM} - M'_{ADM}\Big| \leqslant \frac{1}{16\pi}\lim_{r\to+\infty}\left|\int_{S_r} f\, dvol_{\bar{g}} - \int_{S'_r} f\, dvol_{\bar{g}}\right| \lesssim \lim_{r\to+\infty}\int_{D_r}|\nabla_x f|\, dx \lesssim \lim_{r\to+\infty}\Big(\sup_{D_r}|\nabla_x f|\cdot vol(D_r)\Big) = 0$$

(b)

Using Kruskal coordinates from our theory, we know that the maximally extended Schwarzschild spacetime has two connected unbounded asymptotically flat components, each of which can be covered by Boyer–Lindquist coordinates

$$g = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\sin\theta d\theta^2 + r^2 d\phi^2 \tag{5}$$

The induced metric on $\{t=0\}$ is

$$\begin{aligned}\bar{g} &= \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\sin\theta d\theta^2 + r^2 d\phi^2 \\ &= dr^2 + r^2\sin\theta d\theta^2 + r^2 d\phi^2 \ + \ \frac{2M}{r-2M}dr^2\end{aligned}$$

To verify its asymptotically flatness and to compute its M_{ADM} mass using the definition of the assumption, let us introduce the standard Cartesian coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$
$$r = \sqrt{x^2+y^2+z^2}.$$

Note that

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dr.$$

Then, the induced metric takes the form

$$\bar{g} = dx^2 + dy^2 + dz^2 + \frac{2M}{r-2M}dr^2$$

For convenience, let us denote by $(x_1,x_2,x_3) \doteq (x,y,z)$, and using $dr = \frac{x_1}{r}dx_1 + \frac{x_2}{r}dx_2 + \frac{x_3}{r}dx_3$, the components of the line element can be easily verified to be

$$g_{ij} = \delta_{ij} + \frac{2M}{r-2M}\frac{x_i \cdot x_j}{r^2}, \qquad\qquad r = \sqrt{x_1^2+x_2^2+x_3^2}$$

Note,

$$g_{ij} - \delta_{ij} = \frac{2M}{r-2M}\frac{x_i \cdot x_j}{r^2} \qquad\qquad \text{for all} \ \ r > 2M$$

from which the asymptotic flatness follows (note that the function on the right hand side is of size $O(r^{-1})$, and each ∂_{x^i} derivative improves its decay by an order of r^{-1}).

Next, to compute the M_{ADM} mass, for any $i \neq j$ we have

$$\begin{aligned}\partial_j g_{ij} &= \frac{2M}{r-2M}\frac{x_i}{r^4}\left[r^2 - 3x_j^2 - \frac{2M}{r-2M}x_j^2\right] \\ \partial_i g_{jj} &= -\frac{2M}{r-2M}\frac{x_i}{r^4}x_j^2\left[3 + \frac{2M}{r-2M}\right]\end{aligned}$$

from which we readily get

$$\partial_j g_{ij} - \partial_i g_{jj} = \frac{2M}{r-2M}\frac{x_i}{r^2}$$

Note, we get no contribution when $i=j$ since the difference is zero. Also, the Euclidean unit normal vector N to S_r is given in these coordinates by $N = \dfrac{1}{r}(x_1,x_2,x_3)$, thus we have

$$\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3}(\partial_j g_{ij} - \partial_i g_{jj})N^i \ = \ \sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3}\frac{2M}{r-2M}\frac{x_i^2}{r^3} \ = \ \frac{4M}{r-2M}\frac{1}{r} \ = \ \frac{4M}{1-\frac{2M}{r}}\frac{1}{r^2} \tag{6}$$

Therefore, we readily obtain

$$\begin{aligned} M_{ADM} &= \frac{1}{16\pi}\lim_{r\to\infty}\int_{S_r}\sum_{i=1}^{3}\sum_{\substack{j=1\\ j\neq i}}^{3}(\partial_j g_{ij} - \partial_i g_{jj})N^i dA \\ &= \frac{1}{16\pi}\lim_{r\to\infty}\left(\frac{4M}{1-\frac{2M}{r}}\frac{1}{r^2}\int_{S_r} dA\right) \\ &= \frac{1}{16\pi}\lim_{r\to\infty}\left(\frac{4M}{1-\frac{2M}{r}}\frac{1}{r^2}4\pi r^2\right) \\ &= M \end{aligned}$$

(c)

Following the hint, we need to linearize the Hamiltonian constraint equation which, in view of $\kappa^{(\epsilon)} = 0$, reduces to

$$\bar{R}^{(\epsilon)} = 16\pi\epsilon T(\hat{n},\hat{n})$$

To linearize the scalar curvature we need to express it in terms of metric coefficients using the standard expressions

$$\begin{aligned} {\Gamma_{ab}}^c &= \frac{1}{2}\sum_{d=1}^{n}\left(\frac{\partial\bar{g}^{(\epsilon)}_{bd}}{\partial x^a} + \frac{\partial\bar{g}^{(\epsilon)}_{ad}}{\partial x^b} - \frac{\partial\bar{g}^{(\epsilon)}_{ab}}{\partial x^d}\right)\bar{g}^{(\epsilon)cd} \\ \bar{R}^{(\epsilon)}_{ij} &= \sum_{a=1}^{n}\frac{\partial\Gamma^a_{ij}}{\partial x^a} - \sum_{a=1}^{n}\frac{\partial\Gamma^a_{ai}}{\partial x^j} + \sum_{a=1}^{n}\sum_{b=1}^{n}\left(\Gamma^a_{ab}\Gamma^b_{ij} - \Gamma^a_{ib}\Gamma^b_{aj}\right) \end{aligned}$$

Let us fix normal coordinates (x^1, x^2, x^3) such that $\Gamma^k_{ij}(p) = 0$ for all $i, j, k \in \{1, 2, 3\}$, then we have

$$\begin{aligned} \bar{Ric}^{(\epsilon)}_{ij} = &\sum_{a=1}^{3}\frac{1}{2}\partial_a(\bar{g}^{(\epsilon)ak})\left[\partial_i\bar{g}^{(\epsilon)}_{kj} + \partial_j\bar{g}^{(\epsilon)}_{ki} - \partial_k\bar{g}^{(\epsilon)}_{ij}\right] + \sum_{a=1}^{3}\frac{1}{2}\bar{g}^{(\epsilon)ak}\left[\partial_a\partial_i\bar{g}^{(\epsilon)}_{kj} + \partial_a\partial_j\bar{g}^{(\epsilon)}_{ki} - \partial_a\partial_k\bar{g}^{(\epsilon)}_{ij}\right] \\ &- \sum_{a=1}^{3}\frac{1}{2}\partial_j(\bar{g}^{(\epsilon)a\lambda})\left[\partial_a\bar{g}^{(\epsilon)}_{\lambda j} + \partial_i\bar{g}^{(\epsilon)}_{\lambda a} - \partial_\lambda\bar{g}^{(\epsilon)}_{ai}\right] - \sum_{a=1}^{3}\frac{1}{2}\bar{g}^{(\epsilon)a\lambda}\left[\partial_j\partial_a\bar{g}^{(\epsilon)}_{\lambda i} + \partial_j\partial_i\bar{g}^{(\epsilon)}_{\lambda a} - \partial_j\partial_\lambda\bar{g}^{(\epsilon)}_{ai}\right] \end{aligned}$$

Upon linearization around $\epsilon = 0$ of the above, if $\frac{d}{d\epsilon}$ fall on the first and third term they will vanish (why?). Note,

$$\frac{d}{d\epsilon}\bar{R}^{(\epsilon)}\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left({\bar{g}^{(\epsilon)}}^{ij}\bar{Ric}^{(\epsilon)}_{ij}\right)\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\bar{g}^{(\epsilon)ij}\Big|_{\epsilon=0}\cdot \underbrace{\bar{Ric}^{(\epsilon=0)}_{ij}}_{0} + \delta^{ij}\cdot\frac{d}{d\epsilon}{\bar{Ric}_{ij}}^{(\epsilon)}\Big|_{\epsilon=0}$$

Thus, going back to the previous expression, the only terms that survive are the second and the last, when the derivative $\frac{d}{d\epsilon}$ falls on the bracket terms, for which we get

$$\begin{aligned}
\frac{d}{d\epsilon}\bar{R}^{(\epsilon)}\Big|_{\epsilon=0} &= \delta^{ij}\cdot\frac{d}{d\epsilon}\bar{Ric_{ij}}^{(\epsilon)}\Big|_{\epsilon=0} \\
&= \quad \sum_{i=1}^{3}\sum_{a=1}^{3}\frac{1}{2}\delta^{ij}\delta^{ak}\left[\partial_a\partial_i h_{kj}+\partial_a\partial_j h_{ki}-\partial_a\partial_k h_{ij}\right] \\
&\quad -\sum_{i=1}^{3}\sum_{a=1}^{3}\frac{1}{2}\delta^{ij}\delta^{a\lambda}\left[\partial_j\partial_a h_{\lambda i}+\partial_j\partial_i h_{\lambda a}-\partial_j\partial_\lambda h_{ai}\right] \\
&= \quad \sum_{i=1}^{3}\sum_{a=1}^{3}\frac{1}{2}\left[2\partial_a\partial_i h_{ai}-\partial_a^2 h_{ii}-\cancel{\partial_i\partial_a h_{ai}}-\partial_i^2 h_{aa}+\cancel{\partial_i\partial_a h_{ai}}\right] \\
&\quad \sum_{i=1}^{3}\sum_{a=1}^{3}\left(\partial_a\partial_i h_{ai}-\partial_a^2 h_{ii}\right).
\end{aligned}$$

Hence, linearizing the Hamiltonian constrain equation yields indeed the statement of the assumption.

To show that the M_{ADM} is non-decreasing function of ϵ, we observe that the relation we proved above can simply be written as

$$\begin{aligned}
\sum_{i,j=1}^{3}\left(-\partial_i^2 h_{jj}+\partial_i\partial_j h_{ij}\right) &= 16\pi T(\hat{n},\hat{n})\geqslant 0 \\
\Rightarrow \quad \operatorname{div}\left(\sum_{j=1}^{3}-\partial_i h_{jj}+\partial_j h_{ih}\right) &\geqslant 0
\end{aligned}$$

Integrating on balls of radius r, B_r, and using the divergence theorem yields for any r

$$\int_{S_r}\left(\sum_{j=1}^{3}-\partial_i h_{jj}+\partial_j h_{ih}\right)N^i dA=\int_{B_r}\operatorname{div}\left(\sum_{j=1}^{3}-\partial_i h_{jj}+\partial_j h_{ih}\right)\geqslant 0$$

Note, after taking the $\lim_{r\to\infty}$ of the latter, the first term is simply a positive multiple of $\frac{dM_{ADM}^{(\epsilon)}}{d\epsilon}\Big|_{\epsilon=0}$, which concludes the proof.